Measure and Integration, Oral exam, 03/11/2014

(1) Let Ω be an infinite set and let (ω_n) be a sequence of distinct elements of Ω. Let (a_n) be a sequence of positive numbers. Define $\psi_n : P(\Omega) \to \mathbb{R}$ by

$$\psi_n(E) = \begin{cases} 1, & \omega_n \in E, \\ 0, & \omega_n \notin E, \end{cases} \qquad E \in \Omega.$$

Define $\mu : P(\Omega) \to [0, \infty]$ by

$$\mu(E) = \sum_{n=1}^{\infty} a_n \psi_n(E), \quad E \subset \Omega.$$

Show that μ is a measure on $P(\Omega)$.

(2) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f : \Omega \to \mathbb{R}$ be a measurable function such that

$$\{\omega \in \Omega : f(\omega) > 0\}$$

has positive measure. Show that there exists $N \in \mathbb{N}$ such that the set

$$\{\omega \in \Omega : f(\omega) > 1/N\}.$$

has positive measure.

(3) Let $f \in \mathcal{L}(\Omega, \mathcal{A}, \mathbb{R})$ and assume that for all $A \in \mathcal{A}$

$$\int_A f(\omega)\, d\mu = 0.$$

Show that $f = 0$ almost everywhere.
Hint: use previous exercise.

(4) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let f_n, $n \in \mathbb{N}$, be nonnegative measurable numerical functions on Ω. Show that

$$\limsup_{n\to\infty} \int_\Omega f_n\, d\mu \le \int_\Omega (\limsup_{n\to\infty} f_n)\, d\mu.$$

(5) Calculate

$$\lim_{n\to\infty} \int_{\mathbb{R}} e^{-nx^2+x}\, dx.$$

(6) Show via the dominated convergence theorem that for $a, b > 0$

$$\int_0^1 \frac{t^{a-1}}{1+t^b}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{nb+a}.$$

(7) Let $f \in \mathcal{L}(\Omega, \mathcal{A}, \mu)$ and let M_n be mutually disjoint measurable sets and let $M = \bigcup_{n=1}^{\infty} M_n$. Show that

$$\int_M f\, d\mu = \sum_{n=1}^{\infty} \int_{M_n} f\, d\mu.$$

(8) Calculate the integral

$$\int_0^{\pi/2} \frac{\log(1+\sin\varphi)}{\sin\varphi}\, d\varphi,$$

via

$$\int_0^{\pi/2} \frac{\log(1 + a\sin\varphi)}{\sin\varphi}\, d\varphi.$$

Hint: Differentiate under the integral sign. Use a suitable substitution for integrands involving trigonometric functions.

(9) What can be said about the convergence of

$$\int_0^{\infty} \frac{\sin x}{x^p}\, dx$$

for $p \in \mathbb{R}$?

(10) Determine $\int_A f(x,y)\, dxdy$ when

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, y \geq 0\}, \quad f(x,y) = \frac{e^{(1-y)^{3/2}}}{(1+y)^{1/2}}.$$

(11) Let $\mathcal{A}$ be the Borel σ-algebra on $\Omega = [0,\infty)$ and let μ be a σ-finite measure defined on $\mathcal{A}$.

(a) Show that the function $f(x) = \mu([x,\infty))$, $x \geq 0$, is measurable with respect to $\mathcal{A}$.
Hint: What happens when x increases?

(b) Show that

$$\int_0^{\infty} x\, \mu(dx) = \int_0^{\infty} \mu([t,\infty))\, dt,$$

where the integral on the right-hand side is taken with respect to Lebesgue measure.
Hint: Define, for $x, t \geq 0$,

$$g(x,t) = \begin{cases} 1 & \text{if } t \leq x; \\ 0 & \text{otherwise} \end{cases}$$

and note that $x = \int_0^{\infty} g(x,t)\, dt$.

(12) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $1 < p < q < r \leq \infty$. Show that

$$\mathcal{L}^q(\Omega) \subset \mathcal{L}^p(\Omega) + \mathcal{L}^r(\Omega).$$

Hint: Let $f \in \mathcal{L}^q(\Omega)$, $E = \{\omega \in \Omega : |f(\omega)| > 1\}$, and decompose f by $f = f\mathbf{1}_E + f\mathbf{1}_{\Omega\setminus E}$.

(13) Let the functions $f, g \in L^1(\mathbb{R})$ be defined by

$$f = \mathbf{1}_{[-a,a]}, \quad a > 0, \quad \text{and} \quad g = \mathbf{1}_{[-b,b]}, \quad b \geq a.$$

Calculate $f \star g$.